

reader is then advised to start again from the beginning since the greater familiarity acquired is likely to facilitate a better understanding.

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CHAPTER I

FORMULATION OF THE ECONOMIC PROBLEM

1. The Mathematical Method in Economics

1.1. Introductory Remarks

1.1.1. The purpose of this book is to present a discussion of some fundamental questions of economic theory which require a treatment different from that which they have found thus far in the literature. The analysis is concerned with some basic problems arising from a study of economic behavior which have been the center of attention of economists for a long time. They have their origin in the attempts to find an exact description of the endeavor of the individual to obtain a maximum of utility, or, in the case of the entrepreneur, a maximum of profit. It is well known what considerable—and in fact unsurmounted—difficulties this task involves given even a limited number of typical situations, as, for example, in the case of the exchange of goods, direct or indirect, between two or more persons, of bilateral monopoly, of duopoly, of oligopoly, and of free competition. It will be made clear that the structure of these problems, familiar to every student of economics, is in many respects quite different from the way in which they are conceived at the present time. It will appear, furthermore, that their exact positing and subsequent solution can only be achieved with the aid of mathematical methods which diverge considerably from the techniques applied by older or by contemporary mathematical economists.

1.1.2. Our considerations will lead to the application of the mathematical theory of "games of strategy" developed by one of us in several successive stages in 1928 and 1940–1941.¹ After the presentation of this theory, its application to economic problems in the sense indicated above will be undertaken. It will appear that it provides a new approach to a number of economic questions as yet unsettled.

We shall first have to find in which way this theory of games can be brought into relationship with economic theory, and what their common elements are. This can be done best by stating briefly the nature of some fundamental economic problems so that the common elements will be seen clearly. It will then become apparent that there is not only nothing artificial in establishing this relationship but that on the contrary this

¹ The first phases of this work were published: *J. von Neumann*, "Zur Theorie der Gesellschaftsspiele," *Math. Annalen*, vol. 100 (1928), pp. 295–320. The subsequent completion of the theory, as well as the more detailed elaboration of the considerations of loc. cit. above, are published here for the first time.

theory of games of strategy is the proper instrument with which to develop a theory of economic behavior.

One would misunderstand the intent of our discussions by interpreting them as merely pointing out an analogy between these two spheres. We hope to establish satisfactorily, after developing a few plausible schematizations, that the typical problems of economic behavior become strictly identical with the mathematical notions of suitable games of strategy.

1.2. Difficulties of the Application of the Mathematical Method

→ 1.2.1. It may be opportune to begin with some remarks concerning the nature of economic theory and to discuss briefly the question of the role which mathematics may take in its development.

First let us be aware that there exists at present no universal system of economic theory and that, if one should ever be developed, it will very probably not be during our lifetime. The reason for this is simply that economics is far too difficult a science to permit its construction rapidly, especially in view of the very limited knowledge and imperfect description of the facts with which economists are dealing. Only those who fail to appreciate this condition are likely to attempt the construction of universal systems. Even in sciences which are far more advanced than economics, like physics, there is no universal system available at present.

To continue the simile with physics: It happens occasionally that a particular physical theory appears to provide the basis for a universal system, but in all instances up to the present time this appearance has not lasted more than a decade at best. The everyday work of the research physicist is certainly not involved with such high aims, but rather is concerned with special problems which are "mature." There would probably be no progress at all in physics if a serious attempt were made to enforce that super-standard. The physicist works on individual problems, some of great practical significance, others of less. Unifications of fields which were formerly divided and far apart may alternate with this type of work. However, such fortunate occurrences are rare and happen only after each field has been thoroughly explored. Considering the fact that economics is much more difficult, much less understood, and undoubtedly in a much earlier stage of its evolution as a science than physics, one should clearly not expect more than a development of the above type in economics either.

Second we have to notice that the differences in scientific questions make it necessary to employ varying methods which may afterwards have to be discarded if better ones offer themselves. This has a double implication: In some branches of economics the most fruitful work may be that of careful, patient description; indeed this may be by far the largest domain for the present and for some time to come. In others it may be possible to develop already a theory in a strict manner, and for that purpose the use of mathematics may be required.

Mathematics has actually been used in economic theory, perhaps even in an exaggerated manner. In any case its use has not been highly successful. This is contrary to what one observes in other sciences: There mathematics has been applied with great success, and most sciences could hardly get along without it. Yet the explanation for this phenomenon is fairly simple.

1.2.2. It is not that there exists any fundamental reason why mathematics should not be used in economics. The arguments often heard that because of the human element, of the psychological factors etc., or because there is—allegedly—no measurement of important factors, mathematics will find no application, can all be dismissed as utterly mistaken. Almost all these objections have been made, or might have been made, many centuries ago in fields where mathematics is now the chief instrument of analysis. This "might have been" is meant in the following sense: Let us try to imagine ourselves in the period which preceded the mathematical or almost mathematical phase of the development in physics, that is the 16th century, or in chemistry and biology, that is the 18th century. Taking for granted the skeptical attitude of those who object to mathematical economics in principle, the outlook in the physical and biological sciences at these early periods can hardly have been better than that in economics—*mutatis mutandis*—at present.

→ As to the lack of measurement of the most important factors, the example of the theory of heat is most instructive; before the development of the mathematical theory the possibilities of quantitative measurements were less favorable there than they are now in economics. The precise measurements of the quantity and quality of heat (energy and temperature) were the outcome and not the antecedents of the mathematical theory. This ought to be contrasted with the fact that the quantitative and exact notions of prices, money and the rate of interest were already developed centuries ago.

A further group of objections against quantitative measurements in economics, centers around the lack of indefinite divisibility of economic quantities. This is supposedly incompatible with the use of the infinitesimal calculus and hence (!) of mathematics. It is hard to see how such objections can be maintained in view of the atomic theories in physics and chemistry, the theory of quanta in electrodynamics, etc., and the notorious and continued success of mathematical analysis within these disciplines.

At this point it is appropriate to mention another familiar argument of economic literature which may be revived as an objection against the mathematical procedure.

1.2.3. In order to elucidate the conceptions which we are applying to economics, we have given and may give again some illustrations from physics. There are many social scientists who object to the drawing of such parallels on various grounds, among which is generally found the assertion that economic theory cannot be modeled after physics since it is a

science of social, of human phenomena, has to take psychology into account, etc. Such statements are at least premature. It is without doubt reasonable to discover what has led to progress in other sciences, and to investigate whether the application of the same principles may not lead to progress in economics also. Should the need for the application of different principles arise, it could be revealed only in the course of the actual development of economic theory. This would itself constitute a major revolution. But since most assuredly we have not yet reached such a state—and it is by no means certain that there ever will be need for entirely different scientific principles—it would be very unwise to consider anything else than the pursuit of our problems in the manner which has resulted in the establishment of physical science.

1.2.4. The reason why mathematics has not been more successful in economics must, consequently, be found elsewhere. The lack of real success is largely due to a combination of unfavorable circumstances, some of which can be removed gradually. To begin with, the economic problems were not formulated clearly and are often stated in such vague terms as to make mathematical treatment *a priori* appear hopeless because it is quite uncertain what the problems really are. There is no point in using exact methods where there is no clarity in the concepts and issues to which they are to be applied. Consequently the initial task is to clarify the knowledge of the matter by further careful descriptive work. But even in those parts of economics where the descriptive problem has been handled more satisfactorily, mathematical tools have seldom been used appropriately. They were either inadequately handled, as in the attempts to determine a general economic equilibrium by the mere counting of numbers of equations and unknowns, or they led to mere translations from a literary form of expression into symbols, without any subsequent mathematical analysis.

Next, the empirical background of economic science is definitely inadequate. Our knowledge of the relevant facts of economics is incomparably smaller than that commanded in physics at the time when the mathematization of that subject was achieved. Indeed, the decisive break which came in physics in the seventeenth century, specifically in the field of mechanics, was possible only because of previous developments in astronomy. It was backed by several millennia of systematic, scientific, astronomical observation, culminating in an observer of unparalleled caliber, Tycho de Brahe. Nothing of this sort has occurred in economic science. It would have been absurd in physics to expect Kepler and Newton without Tycho,—and there is no reason to hope for an easier development in economics.

These obvious comments should not be construed, of course, as a disparagement of statistical-economic research which holds the real promise of progress in the proper direction.

It is due to the combination of the above mentioned circumstances that mathematical economics has not achieved very much. The underlying

vagueness and ignorance has not been dispelled by the inadequate and inappropriate use of a powerful instrument that is very difficult to handle.

In the light of these remarks we may describe our own position as follows: The aim of this book lies not in the direction of empirical research. The advancement of that side of economic science, on anything like the scale which was recognized above as necessary, is clearly a task of vast proportions. It may be hoped that as a result of the improvements of scientific technique and of experience gained in other fields, the development of descriptive economics will not take as much time as the comparison with astronomy would suggest. But in any case the task seems to transcend the limits of any individually planned program.

We shall attempt to utilize only some commonplace experience concerning human behavior which lends itself to mathematical treatment and which is of economic importance.

We believe that the possibility of a mathematical treatment of these phenomena refutes the “fundamental” objections referred to in 1.2.2.

It will be seen, however, that this process of mathematization is not at all obvious. Indeed, the objections mentioned above may have their roots partly in the rather obvious difficulties of any direct mathematical approach. We shall find it necessary to draw upon techniques of mathematics which have not been used heretofore in mathematical economics, and it is quite possible that further study may result in the future in the creation of new mathematical disciplines.

To conclude, we may also observe that part of the feeling of dissatisfaction with the mathematical treatment of economic theory derives largely from the fact that frequently one is offered not proofs but mere assertions which are really no better than the same assertions given in literary form. Very frequently the proofs are lacking because a mathematical treatment has been attempted of fields which are so vast and so complicated that for a long time to come—until much more empirical knowledge is acquired—there is hardly any reason at all to expect progress *more mathematico*. The fact that these fields have been attacked in this way—as for example the theory of economic fluctuations, the time structure of production, etc.—indicates how much the attendant difficulties are being underestimated. They are enormous and we are now in no way equipped for them.

1.2.5. We have referred to the nature and the possibilities of those changes in mathematical technique—in fact, in mathematics itself—which a successful application of mathematics to a new subject may produce. It is important to visualize these in their proper perspective.

It must not be forgotten that these changes may be very considerable. The decisive phase of the application of mathematics to physics—Newton's creation of a rational discipline of mechanics—brought about, and can hardly be separated from, the discovery of the infinitesimal calculus. (There are several other examples, but none stronger than this.)

The importance of the social phenomena, the wealth and multiplicity of their manifestations, and the complexity of their structure, are at least equal to those in physics. It is therefore to be expected—or feared—that mathematical discoveries of a stature comparable to that of calculus will be needed in order to produce decisive success in this field. (Incidentally, it is in this spirit that our present efforts must be discounted.) *A fortiori* it is unlikely that a mere repetition of the tricks which served us so well in physics will do for the social phenomena too. The probability is very slim indeed, since it will be shown that we encounter in our discussions some mathematical problems which are quite different from those which occur in physical science.

These observations should be remembered in connection with the current overemphasis on the use of calculus, differential equations, etc., as the main tools of mathematical economics.

1.3. Necessary Limitations of the Objectives

1.3.1. We have to return, therefore, to the position indicated earlier: It is necessary to begin with those problems which are described clearly, even if they should not be as important from any other point of view. It should be added, moreover, that a treatment of these manageable problems may lead to results which are already fairly well known, but the exact proofs may nevertheless be lacking. Before they have been given the respective theory simply does not exist as a scientific theory. The movements of the planets were known long before their courses had been calculated and explained by Newton's theory, and the same applies in many smaller and less dramatic instances. And similarly in economic theory, certain results—say the indeterminateness of bilateral monopoly—may be known already. Yet it is of interest to derive them again from an exact theory. The same could and should be said concerning practically all established economic theorems.

1.3.2. It might be added finally that we do not propose to raise the question of the practical significance of the problems treated. This falls in line with what was said above about the selection of fields for theory. The situation is not different here from that in other sciences. There too the most important questions from a practical point of view may have been completely out of reach during long and fruitful periods of their development. This is certainly still the case in economics, where it is of utmost importance to know how to stabilize employment, how to increase the national income, or how to distribute it adequately. Nobody can really answer these questions, and we need not concern ourselves with the pretension that there can be scientific answers at present.

The great progress in every science came when, in the study of problems which were modest as compared with ultimate aims, methods were developed which could be extended further and further. The free fall is a very trivial physical phenomenon, but it was the study of this exceedingly simple

fact and its comparison with the astronomical material, which brought forth mechanics.

It seems to us that the same standard of modesty should be applied in economics. It is futile to try to explain—and “systematically” at that—everything economic. The sound procedure is to obtain first utmost precision and mastery in a limited field, and then to proceed to another, somewhat wider one, and so on. This would also do away with the unhealthy practice of applying so-called theories to economic or social reform where they are in no way useful.

We believe that it is necessary to know as much as possible about the behavior of the individual and about the simplest forms of exchange. This standpoint was actually adopted with remarkable success by the founders of the marginal utility school, but nevertheless it is not generally accepted. Economists frequently point to much larger, more “burning” questions, and brush everything aside which prevents them from making statements about these. The experience of more advanced sciences, for example physics, indicates that this impatience merely delays progress, including that of the treatment of the “burning” questions. There is no reason to assume the existence of shortcuts.

1.4. Concluding Remarks

1.4. It is essential to realize that economists can expect no easier fate than that which befell scientists in other disciplines. It seems reasonable to expect that they will have to take up first problems contained in the very simplest facts of economic life and try to establish theories which explain them and which really conform to rigorous scientific standards. We can have enough confidence that from then on the science of economics will grow further, gradually comprising matters of more vital importance than those with which one has to begin.¹

The field covered in this book is very limited, and we approach it in this sense of modesty. We do not worry at all if the results of our study conform with views gained recently or held for a long time, for what is important is the gradual development of a theory, based on a careful analysis of the ordinary everyday interpretation of economic facts. This preliminary stage is necessarily *heuristic*, i.e. the phase of transition from unmathematical plausibility considerations to the formal procedure of mathematics. The theory finally obtained must be mathematically rigorous and conceptually general. Its first applications are necessarily to elementary problems where the result has never been in doubt and no theory is actually required. At this early stage the application serves to corroborate the theory. The next stage develops when the theory is applied

¹ The beginning is actually of a certain significance, because the forms of exchange between a few individuals are the same as those observed on some of the most important markets of modern industry, or in the case of barter exchange between states in international trade.

to somewhat more complicated situations in which it may already lead to a certain extent beyond the obvious and the familiar. Here theory and application corroborate each other mutually. Beyond this lies the field of real success: genuine prediction by theory. It is well known that all mathematized sciences have gone through these successive phases of evolution.

2. Qualitative Discussion of the Problem of Rational Behavior

2.1. The Problem of Rational Behavior

2.1.1. The subject matter of economic theory is the very complicated mechanism of prices and production, and of the gaining and spending of incomes. In the course of the development of economics it has been found, and it is now well-nigh universally agreed, that an approach to this vast problem is gained by the analysis of the behavior of the individuals which constitute the economic community. This analysis has been pushed fairly far in many respects, and while there still exists much disagreement the significance of the approach cannot be doubted, no matter how great its difficulties may be. The obstacles are indeed considerable, even if the investigation should at first be limited to conditions of economics statics, as they well must be. One of the chief difficulties lies in properly describing the assumptions which have to be made about the motives of the individual. This problem has been stated traditionally by assuming that the consumer desires to obtain a maximum of utility or satisfaction and the entrepreneur a maximum of profits.

The conceptual and practical difficulties of the notion of utility, and particularly of the attempts to describe it as a number, are well known and their treatment is not among the primary objectives of this work. We shall nevertheless be forced to discuss them in some instances, in particular in 3.3. and 3.5. Let it be said at once that the standpoint of the present book on this very important and very interesting question will be mainly opportunistic. We wish to concentrate on one problem—which is not that of the measurement of utilities and of preferences—and we shall therefore attempt to simplify all other characteristics as far as reasonably possible. We shall therefore assume that the aim of all participants in the economic system, consumers as well as entrepreneurs, is money, or equivalently a single monetary commodity. This is supposed to be unrestrictedly divisible and substitutable, freely transferable and identical, even in the quantitative sense, with whatever “satisfaction” or “utility” is desired by each participant. (For the quantitative character of utility, cf. 3.3. quoted above.)

It is sometimes claimed in economic literature that discussions of the notions of utility and preference are altogether unnecessary, since these are purely verbal definitions with no empirically observable consequences, i.e., entirely tautological. It does not seem to us that these notions are qualitatively inferior to certain well established and indispensable notions in

physics, like force, mass, charge, etc. That is, while they are in their immediate form merely definitions, they become subject to empirical control through the theories which are built upon them—and in no other way. Thus the notion of utility is raised above the status of a tautology by such economic theories as make use of it and the results of which can be compared with experience or at least with common sense.

2.1.2. The individual who attempts to obtain these respective maxima is also said to act “rationally.” But it may safely be stated that there exists, at present, no satisfactory treatment of the question of rational behavior. There may, for example, exist several ways by which to reach the optimum position; they may depend upon the knowledge and understanding which the individual has and upon the paths of action open to him. A study of all these questions in qualitative terms will not exhaust them, because they imply, as must be evident, quantitative relationships. It would, therefore, be necessary to formulate them in quantitative terms so that all the elements of the qualitative description are taken into consideration. This is an exceedingly difficult task, and we can safely say that it has not been accomplished in the extensive literature about the topic. The chief reason for this lies, no doubt, in the failure to develop and apply suitable mathematical methods to the problem; this would have revealed that the maximum problem which is supposed to correspond to the notion of rationality is not at all formulated in an unambiguous way. Indeed, a more exhaustive analysis (to be given in 4.3.-4.5.) reveals that the significant relationships are much more complicated than the popular and the “philosophical” use of the word “rational” indicates.

A valuable qualitative preliminary description of the behavior of the individual is offered by the Austrian School, particularly in analyzing the economy of the isolated “Robinson Crusoe.” We may have occasion to note also some considerations of Böhm-Bawerk concerning the exchange between two or more persons. The more recent exposition of the theory of the individual's choices in the form of indifference curve analysis builds up on the very same facts or alleged facts but uses a method which is often held to be superior in many ways. Concerning this we refer to the discussions in 2.1.1. and 3.3.

We hope, however, to obtain a real understanding of the problem of exchange by studying it from an altogether different angle; this is, from the perspective of a “game of strategy.” Our approach will become clear presently, especially after some ideas which have been advanced, say by Böhm-Bawerk—whose views may be considered only as a prototype of this theory—are given correct quantitative formulation.

2.2. “Robinson Crusoe” Economy and Social Exchange Economy

2.2.1. Let us look more closely at the type of economy which is represented by the “Robinson Crusoe” model, that is an economy of an isolated single person or otherwise organized under a single will. This economy is

all those of $\mathfrak{B}$ —which are not empty. This again is clearly a partition, the *superposition* of $\mathfrak{A}$, $\mathfrak{B}$.¹

Finally, we also define the above relations for two partitions $\mathfrak{A}$, $\mathfrak{B}$ within a given set C .

- (8:B:e) $\mathfrak{A}$ is a *subpartition* of $\mathfrak{B}$ within C , if every A belonging to $\mathfrak{A}$ which is a subset of C is also subset of some B belonging to $\mathfrak{B}$ which is a subset of C .
- (8:B:f) $\mathfrak{A}$ is *equal* to $\mathfrak{B}$ within C if the same subsets of C are elements of $\mathfrak{A}$ and of $\mathfrak{B}$.

Clearly footnote 3 on p. 63 applies again, *mutatis mutandis*. Also, the above concepts within Ω are the same as the original unqualified ones.

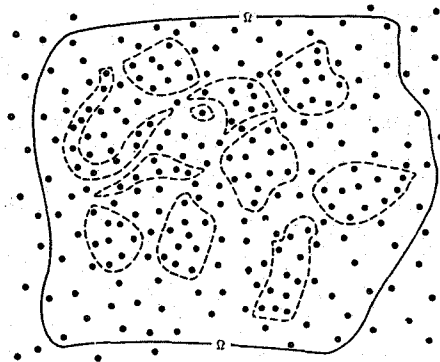


Figure 4.

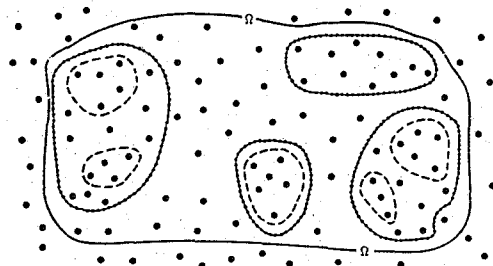


Figure 5.

8.3.2. We give again some graphical illustrations, in the sense of 8.2.3.

We begin by picturing a partition. We shall not give the elements of the partition—which are sets—names, but denote each one by an encircling line — — — (Figure 4).

We picture next two partitions $\mathfrak{A}$, $\mathfrak{B}$ distinguishing them by marking the encircling lines of the elements of $\mathfrak{A}$ by — — — and of the elements of $\mathfrak{B}$ by — — —

¹ It is easy to show that the superposition of $\mathfrak{A}$, $\mathfrak{B}$ is a subpartition of both $\mathfrak{A}$ and $\mathfrak{B}$ —and that every partition $\mathfrak{C}$ which is a subpartition of both $\mathfrak{A}$ and $\mathfrak{B}$ is also one of their superposition. Hence the name. Cf. G. Birkhoff, loc. cit. Chapt. I-II.

(Figure 6). In this figure one neither $\mathfrak{A}$ is a subpartition $\mathfrak{B}$ nor is $\mathfrak{B}$ one of $\mathfrak{A}$ (Figure 6). We leave it to the reader to determine the superposition of $\mathfrak{A}$, $\mathfrak{B}$ in this figure.

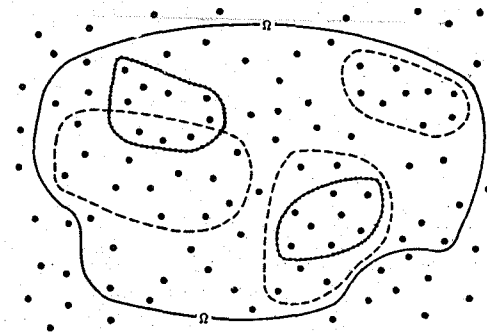


Figure 6.

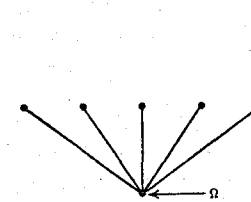


Figure 7.

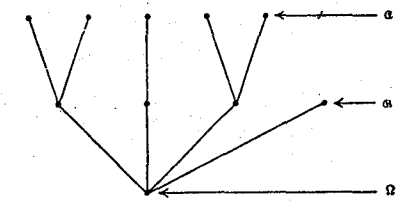


Figure 8.

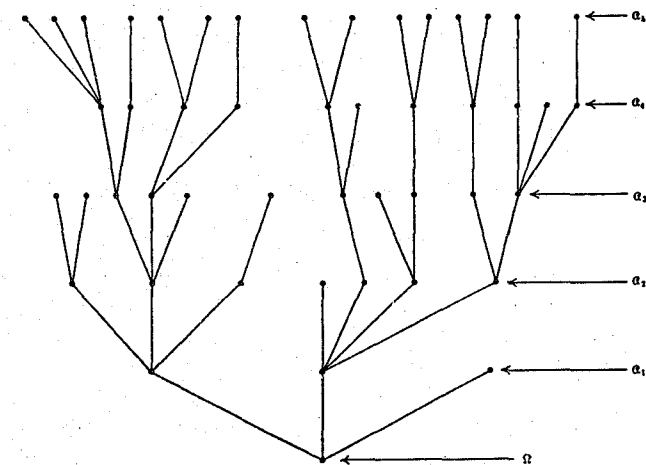


Figure 9.

Another, more schematic, representation of partitions obtains by representing the set Ω by one dot, and every element of the partition—which is a